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Chebyshev Approximation Constants Related to Entire Functions of Perfectly Regular Growth

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Certain entire functions are studied for Chebyshev rational approximations on the positive real axis. It is shown that each function of this class has a geometric convergence property.

1. INTRODUCTION

Let π_m denote the collection of all real polynomials of degree at most m and $\pi_{m,n}$ the collection of all real rational functions $r_{m,n}(x) \equiv p_m(x)/q_n(x)$, $p_m \in \pi_m$, $q_n \in \pi_n$. Let $f(z)$ be an entire function $\sum_{k=0}^{\infty} a_k z^k \not\equiv 0$ with nonnegative a_k, and let

$$\lambda_{m,n} = \inf_{\pi_{m,n}} \sup_{0<x<\infty} \left| \frac{1}{f(x)} - r_{m,n}(x) \right|$$

denote the Chebyshev constants for $1/f$ in $[0, +\infty)$.

In some recent papers Meinardus and Varga [3], Meinardus, Reddy, Taylor and Varga [4], and others (see [1, 5, 6, 7, 8]) have considered these constants. In [3] Meinardus and Varga proved the following:

THEOREM A. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function of perfectly regular growth order $\rho (0 < \rho < \infty)$, with nonnegative coefficients. Then*

$$\lim_{n\to\infty} \left\{ \sup_{0<x<\infty} \left| \frac{1}{s_n(x)} - \frac{1}{f(x)} \right| \right\}^{1/n} = \frac{1}{2^{1/\rho}},$$

where $s_n(x) = \sum_{k=0}^{n} a_k x^k$.

THEOREM B. *Assume the hypothesis of Theorem A. Then for any sequence $\{m(n)\}_{n=0}^{\infty}$ of nonnegative integers with $m(n) \leqslant n$ for all $n \geqslant 0$*

$$\limsup_{n\to\infty} \{\lambda_{m(n),n}\}^{1/n} \leqslant \frac{1}{2^{1/\rho}}.$$

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THEOREM C. *Assume the hypothesis of Theorem A. Then*

$$\limsup_{n\to\infty}(\lambda_{0,n})^{1/n} \geqslant \frac{1}{2^{2+1/\rho}}.$$

In this paper we place a less restrictive hypothesis on the maximum modulus $M(r, f)$ and obtain extensions of Theorems A, B and C. The proof uses Approximation techniques of Meinardus and Varga [3] with modifications necessary to use a wider class of comparison functions.

THEOREM 1.

(a) *Let $f(z)$ be an entire function with nonnegative coefficients and $f(0) > 0$.*

(b) *Suppose that $f(z)$ is of order $\rho(0 < \rho < \infty)$ and of perfectly regular growth with respect to a proximate order $\rho(r)$, that is*

$$\lim_{r\to\infty}\frac{\log M(r, f)}{r^{\rho(r)}} = 1, \qquad \lim_{r\to\infty}\rho(r) = \rho. \tag{1.1}$$

Let $\rho(r) > 0$ for $r \geqslant x_0$, and let ω be a real valued function defined on $[x_0, \infty)$ by the relation

$$\omega(\rho(x)\, x^{\rho(x)}) = 1/\rho(x).$$

(c) *Assume that $x\omega(x)\{\log x - \log e\rho\}$ is convex on (x_0, ∞).*

Then

$$\lim_{n\to\infty}\left\{\sup_{0\leqslant x<\infty}\left|\frac{1}{s_n(x)} - \frac{1}{f(x)}\right|\right\}^{1/n} = \frac{1}{2^{1/\rho}}, \tag{1.2}$$

where $s_n(x)$ is the nth partial sum of the series $f(x) = \sum_{k=0}^{\infty} a_k x^k$.

$$\frac{1}{2^{2+1/\rho}} \leqslant \limsup_{n\to\infty}\lambda_{0,n}^{1/n} \leqslant \frac{1}{2^{1/\rho}}. \tag{1.3}$$

For any sequence $\{m(n)\}_{n=0}^{\infty}$ of nonnegative integers with $m(n) \leqslant n$ for all $n \geqslant 0$,

$$\limsup_{n\to\infty}(\lambda_{m(n),n})^{1/n} \leqslant \frac{1}{2^{1/\rho}}. \tag{1.4}$$

COROLLARY 1.1. *Assume the hypotheses* (a) *and* (b). *If $\rho''(x)$ exists and is $o(1/x^2 \log x)$ as $x \to \infty$, then the condition* (c) *is satisfied and the conclusions* (1.2), (1.3) *and* (1.4) *hold.*

Let $l_k x$ denothe the kth iterate of the logarithmic function $l_1 x = \log x$.

COROLLARY 1.2. *Assume the hypothesis* (a). *If*

$$\log M(r, f) \sim Ar^{\rho}(\ell_1 r)^{\alpha_1} \cdots (\ell_k r)^{\alpha_k} (A > 0,\ \rho > 0,$$

α's are any real numbers) then the conclusions (1.2), (1.3) *and* (1.4) *hold.*

In the sequel $n > n_0$ (or $x > x_0$) will mean that n (resp. x) is sufficiently large. The value n_0 (or x_0) will in general vary.

2. PROOF OF THEOREM 1

(i) It is known that [9; pp. 209–210] $\omega(x)$ is continuous on $[x_0, \infty)$ and $x = \{y\omega(y)\}^{\omega(y)}$. Further $\omega(x)$ is differentiable for $x > x_0$ except at isolated points at which $\omega'(x-0)$ and $\omega'(x+0)$ exist and satisfy

$$\lim_{x\to\infty} \omega(x) = 1/\rho, \qquad \lim_{x\to\infty} x\omega'(x) \log x = 0. \tag{2.1}$$

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Given $\epsilon > 0$ we have [9] for all $n > n_0(x_0, \epsilon)$,

$$a_n^{1/n} < (1+\epsilon) \left\{\frac{n}{e\rho}\right\}^{-\omega(n)}, \tag{2.2}$$

and there exists a sequence $\{n_p\}$ of strictly increasing positive integers such that $\lim_{p\to\infty} n_{p+1}/n_p = 1$ and (writing $a_{n_p} = a(n_p)$),

$$a(n_p)^{1/n_p} > (1-\epsilon) \left\{\frac{n_p}{e\rho}\right\}^{-\omega(n_p)}, \qquad p = 1, 2, \ldots \tag{2.3}$$

Now, for $n > n_0$,

$$0 \leqslant \frac{1}{s_n(x)} - \frac{1}{f(x)} \leqslant \left\{\sum_{k=n+1}^{\infty} (1+\epsilon)^k \left(\frac{e\rho}{k}\right)^{k\omega(k)} x^k\right\} \frac{1}{s_n(x) f(x)}.$$

Write, for $n > n_0$,

$$X(n) = \frac{1}{1+\epsilon} \left(\frac{n+2}{e\rho}\right)^{(n+2)\omega(n+2)} \left(\frac{e\rho}{n+1}\right)^{(n+1)\omega(n+1)},$$

$$\delta_n = \exp\left(\frac{-n}{\log n}\right),\ T^* = \frac{x}{X(n)} \text{ and let } 0 \leqslant x \leqslant X(n)(1-\delta_n).$$

Using the convexity hypothesis we have

$$\frac{1}{s_n(x)} - \frac{1}{f(x)} \leqslant \left\{(x(1+\epsilon))^{n+1} \left(\frac{e\rho}{n+1}\right)^{(n+1)\omega(n+1)} \frac{1}{1-T^*}\right\} \frac{1}{s_n(x) f(x)}. \tag{2.4}$$

Let n be odd, $n+1=2n_p$. Then

$$\{s_n(x)\}^2 \geqslant a^2(n_p)x^{2n_p}$$

$$\omega(2n_p)-\omega(n_p)=o\left(\frac{1}{\log n_p}\right), \tag{2.5}$$

$$\omega(n+2)-\omega(n+1)=o\left(\frac{1}{n\log n}\right).$$

From (2.3), (2.4) and (2.5) we have for $0\leqslant x\leqslant X(n)(1-\delta_n)$, $n=2n_p-1$, $p>n_0$,

$$\left(\frac{1}{s_n(x)}-\frac{1}{f(x)}\right)^{1/n}\leqslant\left(\frac{1+\epsilon}{1-\epsilon}\right)^2(1+\epsilon)\exp\left\{\frac{-2n_p\omega(n_p)\log 2}{2n_p-1}\right\}. \tag{2.6}$$

If $x>X(n)(1-\delta_n)$, $n=2n_p-1$, we have

$$\left(\frac{1}{s_n(x)}-\frac{1}{f(x)}\right)^{1/n}\leqslant(a(n_p)x^{n_p})^{-1/n}$$

$$\leqslant\exp\left\{\frac{-n_p}{2n_p-1}\left(\log X(n)+\log(1-\delta_n)-\log\frac{1}{1-\epsilon}\right.\right.$$

$$\left.\left.-\,\omega(n_p)(\log n_p-\log e\rho)\right)\right\}. \tag{2.7}$$

Now

$$\log X(n)-\omega(n_p)(\log n_p-\log e\rho)=\log\frac{1}{1+\epsilon}+\omega(n_p)(1+\log 2)+o(1),$$

and

$$\exp\left\{-\frac{1}{2\rho}(1+\log 2)\right\}<\exp\left(\frac{-\log 2}{\rho}\right).$$

Hence we have from (2.1), (2.6) and (2.7), for $x\geqslant 0$

$$\limsup_{\substack{n=2n_p-1\\ p\to\infty}}\left\|\frac{1}{s_n(x)}-\frac{1}{f(x)}\right\|^{1/n}\leqslant\exp\left(\frac{-\log 2}{\rho}\right). \tag{2.8}$$

Now write for $n>n_0$,

$$G_n=\left\|\frac{1}{s_n(x)}-\frac{1}{f(x)}\right\|; \tag{2.9}$$

then since $G_n\downarrow$ and $n_{p+1}\sim n_p$, we get from (2.8) (cf: [3]),

$$\limsup_{n\to\infty}\left\|\frac{1}{s_n(x)}-\frac{1}{f(x)}\right\|^{1/n}\leqslant\exp\left(\frac{-\log 2}{\rho}\right). \tag{2.10}$$

(ii) Let $\phi(t)$ be the unique (for $t > x_0$) solution of the equation

$$\frac{t}{2\rho} = x^{\rho(x)}. \tag{2.11}$$

Then for $t > x_0(\epsilon)$,

$$f(\phi(t)) \leqslant \exp\{(1+\epsilon)\,\phi(t)^{\rho(\phi(t))}\} = \exp\left\{\frac{(1+\epsilon)t}{2\rho}\right\}.$$

Hence for $0 \leqslant x \leqslant \phi(n)$, $n > n_0$,

$$0 \leqslant f(x) \leqslant f(\phi(n)) \leqslant \exp\left\{(1+\epsilon)\frac{n}{2\rho}\right\}.$$

Let q be any positive number such that

$$\limsup_{n\to\infty} (\lambda_{0,n})^{1/n} < \frac{1}{q}. \tag{2.12}$$

From part (i) we can take $\log q > (1+\epsilon)/2\rho$. Hence for $0 \leqslant x \leqslant \phi(n)$, $f(x) < n^q$. Now (2.12) implies that $\lambda_{0,n} \leqslant (1/q)^n$ for $n > n_0$. Hence there exists $\{p_n(x)\}_{n=0}^{\infty}$ with $p_n \in \pi_n$ such that

$$\left\| \frac{1}{p_n(x)} - \frac{1}{f(x)} \right\| \leqslant \frac{1}{q^n}, \qquad n > n_0.$$

This gives, for $0 \leqslant x \leqslant \phi(n)$, $n > n_0$,

$$| p_n(x) - f(x)| \leqslant \exp\left(\frac{n}{\rho}(1+\epsilon)\right) \Big/ \left\{q^n - \exp\left(\frac{n}{2\rho}(1+\epsilon)\right)\right\}. \tag{2.13}$$

Let, for $n \geqslant 0$,

$$K_n = \inf_{r_n \in \pi_n} \sup_{0 \leqslant x \leqslant \phi(n)} | r_n(x) - f(x)|.$$

Then (cf: [3])

$$K_n \geqslant \frac{(\phi(n))^{n+1}}{2^{2n+1}} a_{n+1}.$$

Now, for a sequence $\{n_k\}$, $k \geqslant 1$ [9], $n = n_k$,

$$a(n+1) \geqslant (1-\epsilon)^{n+1} \left\{\frac{e\rho}{n}\right\}^{(n+1)\omega(n+1)} \tag{2.14}$$

Hence we have for $n = n_k$, $n > n_0$,

$$(1-\epsilon)^{n+1} \left\{\frac{e\rho}{n}\right\}^{(n+1)\omega(n+1)}$$
$$\leqslant \left\{2^{2n+1} \exp\left(\frac{n}{\rho}(1+\epsilon)\right)\right\} \Big/ \left[\phi(n)^{n+1} \left\{q^n - \exp\left(\frac{n}{2\rho}(1+\epsilon)\right)\right\}\right]$$

which simplifies to

$$n \log q \leqslant \frac{n}{\rho}(1+\epsilon) - (n+1)\log(1-\epsilon) + (n+1)\,\omega(n+1)\left\{\log \frac{n}{e\rho}\right\}$$
$$+ (2n+1)\log 2 - (n+1)\log \phi(n). \tag{2.15}$$

Let $\psi(T)$ be the unique solution (for $T > x_0$) of the equation

$$\frac{T}{\rho} = x^{\rho(x)}. \tag{2.16}$$

Then for $n = \rho x^{\rho(x)} \equiv \rho x^{\rho} L(x)$ we have $x = \psi(n)$; and for $n > n_0$,

$$\omega(\rho(x)x^{\rho(x)}) = \omega(n) + o\left(\frac{1}{\log \psi(n)} \frac{\log L(\psi(n))}{\log n}\right),$$

and so

$$\begin{aligned} \omega(n) &= \omega(\rho(x)x^{\rho(x)}) + o\left(\frac{1}{\log n}\right) \\ &= \left\{\rho + \frac{\log L(x)}{\log x}\right\}^{-1} + o\left(\frac{1}{\log n}\right). \end{aligned} \tag{2.17}$$

Now

$$\log \psi(n) - \log \phi(n) = \frac{\log 2}{\rho} + o(1),$$

and so

$$\omega(n)(\log n - \log(e\rho)) - \log \phi(n) = \frac{\log 2 - 1}{\rho} + o(1). \tag{2.18}$$

From (2.15) and (2.18) we get

$$\log q \leqslant \frac{1}{\rho} + 2\log 2 + \frac{\log 2 - 1}{\rho} + B\epsilon$$

where $B = B(\rho)$. Since ϵ is arbitrary we get

$$q \leqslant 2^{2+1/\rho},$$

and consequently, from our choice of q,

$$\limsup_{n\to\infty} \lambda_{0,n}^{1/n} \geqslant 1/2^{2+1/\rho}. \tag{2.19}$$

(iii) For $x \leqslant x_0$, $n \geqslant 0$,

$$\frac{1}{s_n(x)} - \frac{1}{f(x)} \geqslant \frac{a_{n+1}x^{n+1}}{\{f(x)\}^2}. \tag{2.20}$$

We take $n = n_p - 1$ and use (2.3). Further, for $x > x_0$,

$$\frac{x^{n+1}}{\{f(x)\}^2} \geqslant \exp\{(n+1)\log x - (2+\epsilon)x^{\rho(x)}\}. \tag{2.21}$$

Let $x = \xi(n)$ $(n > n_0)$ be the unique solution of the equation

$$\frac{n+1}{(2+\epsilon)\rho} = x^{\rho(x)}. \tag{2.22}$$

We evaluate the right side of (2.21) when $x = \xi(n)$, and note that

$$\omega(\rho x^{\rho(x)}) = \frac{1}{\rho(x)} + o\left(\frac{1}{\log x}\right),$$

$$(\log n)\left\{\frac{1}{\rho(\xi(n))} - \omega(n)\right\} = o(1).$$

Hence at the point $x = \xi(n)$, $n = n_p - 1$,

$$\frac{1}{s_n(x)} - \frac{1}{f(x)} \geqslant (1-\epsilon)^{n_p} \exp\left\{\begin{matrix}-n_p\omega(n_p)(\log n_p - \log \epsilon\rho) \\ +n_p \log \xi(n) - n_p/\rho\end{matrix}\right\}.$$

Now $\log \xi(n) = (\log(n+1) - \log(2+\epsilon) - \log \rho)/\rho(\xi(n))$, and so

$$\left\|\frac{1}{s_n(x)} - \frac{1}{f(x)}\right\|^{1/n} \geqslant (1-\epsilon)^{n_p/n_p-1} \exp\left\{\frac{-\log(2+\epsilon)}{\rho} + o(1)\right\}.$$

Hence by (2.9)

$$\liminf_{\substack{n=n_p-1 \\ p\to\infty}} G_n^{1/n} \geqslant \exp\left\{\frac{-\log 2}{\rho}\right\}.$$

Since $G_n \downarrow$ and $n_{p+1} \sim n_p$ we get

$$\liminf_{n\to\infty} G_n^{1/n} \geqslant \exp\left(\frac{-\log 2}{\rho}\right). \tag{2.23}$$

The relation (1.2) follows from (2.10) and (2.23), and (1.3) from (2.10) and (2.19), and (1.4) from (2.10), since $0 \leqslant \lambda_{n,n} \leqslant \lambda_{n-1,n} \leqslant \cdots \leqslant \lambda_{0,n} \leqslant G_n$.

3. Proof of Corollary 1.1

Since $\rho''(x)$ exists (this hypothesis implies that we are working with a smaller class of proximate orders (cf: [2, pp. 39–41]), $\rho'(x)$ exists and is $o(1/x \log x)$. Now if $y = \rho(x)\, x^{\rho(x)}$ and $\omega(y) = 1/\rho(x)$, we get $d\omega/dy =$

$o(1/y \log y)$, $d^2\omega/dy^2 = o(1/y^2 \log y)$, $y \to \infty$. Now let $\xi(x) = x\omega(x)$ $(\log x - \log e\rho)$, then for $x > x_0$, $d^2\xi/dx^2 > 0$. This implies that condition (c) is satisfied.

Proof of Corollary 1.2. Here

$$r^{\rho(r)} = r^\rho L(r) = Ar^\rho(\ell_1 r)^{\alpha_1} \cdots (\ell_k r)^{\alpha_k}.$$

Hence $\rho''(x) = o(1/x^2 \log x)$ and so we use Corollary 1.1 to complete the proof.

4. Geometric convergence

If we are interested in geometric convergence only, that is, in showing that

$$\limsup_{n\to\infty} \lambda_{0,n}^{1/n} < 1, \tag{4.1}$$

then conditions less restrictive than those of Theorem 1 will suffice. We state them, in Theorem 2, and omit the proof of this theorem (cf: [4, pp. 180–182]). Let $L(r)$ be a slowly changing function [2, p. 32].

Theorem 2. *Let $f(z)$ be an entire function with nonnegative coefficients, and $f(0) > 0$, and of finite positive order ρ. If for some slowly changing function $L(r)$,*

$$0 < \liminf_{r\to\infty} \frac{\log M(r, f)}{r^\rho L(r)} \leqslant \limsup_{r\to\infty} \frac{\log (Mr, f)}{r^\rho L(r)} < \infty,$$

then $f(z)$ has the geometric convergence property (4.1).

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